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An Initial Value Method for Eigenvalue Problems Using Compound Matrices

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An initial value method, based on the use of certain compound matrices, is presented for the treatment of eigenvalue problems for stiff ordinary differential equations. The method appears to overcome not only the parasitic growth problems associated with standard shooting methods but also the difficulties, due to the presence of singularities, of the Riccati method. Two examples are given to illustrate, both analytically and numerically, the essential features of the method.

1. INTRODUCTION

Eigenvalue problems for ordinary differential equations are usually treated by first defining a solution matrix which satisfies certain prescribed initial conditions and the required eigenvalues are then obtained as the roots of some minor of the solution matrix. If we attempt to evaluate this minor by computing its elements separately, as in a standard shooting method, then there may be a serious loss of accuracy numerically especially when the differential equation is stiff. This difficulty can be avoided, however, by considering the differential equation satisfied by a certain compound matrix whose elements are the minors of the solution matrix, and in this way we can compute the required minor directly.

Compound matrices have been used by Gilbert and Backus [3] in their discussion of elastic wave problems and they have also been used recently [5] to derive a uniform approximation to the eigenvalue relation for the Orr–Sommerfeld problem. In this paper, therefore, we wish to show how they can be used for eigenvalue problems for linear ordinary differential equations. In Section 2 we give a general outline of the method with emphasis on equations of fourth order and two examples are discussed in Section 3 to illustrate the main features of the method. Finally, in Section 4, we discuss briefly the relationship between the compound matrix method and the one suggested recently by Golberg [4].

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2. Outline of the Method

To illustrate the basic ideas involved, consider the linear fourth-order equation

$$L(\phi) \equiv \phi^{iv} - a_1\phi''' - a_2\phi'' - a_3\phi' - a_4\phi = 0, \tag{1}$$

where a_1, a_2, a_3, and a_4 are functions of x and $0 \leqslant x \leqslant 1$. To be definite, we shall also suppose that the boundary conditions at $x = 0$ are $\phi = \phi' = 0$. The boundary conditions at $x = 1$, however, need not be specified until later.

For the present purposes it is convenient to rewrite (1) as a system of first-order equations. Thus, if we let $\boldsymbol{\phi} = [\phi, \phi', \phi'', \phi''']^T$ then (1) becomes

$$\boldsymbol{\phi}' = \mathbf{A}(x)\boldsymbol{\phi}, \tag{2}$$

where

$$\mathbf{A}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_4 & a_3 & a_2 & a_1 \end{bmatrix}. \tag{3}$$

Now let $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ be two solutions of (1) which satisfy the initial conditions

$$\boldsymbol{\phi}_1(0) = [0, 0, 1, 0]^T \quad \text{and} \quad \boldsymbol{\phi}_2(0) = [0, 0, 0, 1]^T, \tag{4}$$

and consider the 4×2 solution matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \\ \phi_1''' & \phi_2''' \end{bmatrix}. \tag{5}$$

The 2×2 minors of the matrix $\boldsymbol{\Phi}$ are

$$\begin{aligned} y_1 &= \phi_1\phi_2' - \phi_1'\phi_2, & y_4 &= \phi_1'\phi_2'' - \phi_1''\phi_2', \\ y_2 &= \phi_1\phi_2'' - \phi''\phi_2, & y_5 &= \phi_1'\phi_2''' - \phi_1'''\phi_2', \\ y_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, & y_6 &= \phi''\phi_2''' - \phi_1'''\phi_2'', \end{aligned} \tag{6}$$

and they satisfy the quadratic identity [1]

$$y_1y_6 - y_2y_5 + y_3y_4 = 0. \tag{7}$$

By using the general theory given in [3] or by a direct calculation it is easy to show that $\mathbf{y} = [y_1, \ldots, y_6]^T$ satisfies the equation

$$\mathbf{y}' = \mathbf{B}(x)\mathbf{y}, \tag{8}$$

where

$$\mathbf{B}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_4 & 0 & 0 & a_2 & a_1 & 1 \\ 0 & -a_4 & 0 & -a_3 & 0 & a_1 \end{bmatrix}. \tag{9}$$

Thus $\mathbf{y}$ is the second compound of Φ and it satisfies the initial condition

$$\mathbf{y}(0) = [0, 0, 0, 0, 0, 1]^T. \tag{10}$$

The boundary conditions on ϕ at $x = 1$ will imply that some element of $\mathbf{y}$ or, more generally, a linear combination of the elements of $\mathbf{y}$ must vanish there and this condition will provide the required eigenvalue relation. In actual calculations, of course, some iterative procedure must be used to vary the eigenvalue parameter until this condition is satisfied to some prescribed degree of accuracy.

Once the required eigenvalue has been obtained by the method just described, we can then proceed to the determination of the corresponding eigenfunction ϕ (say). Clearly there must exist constants λ and μ such that

$$\begin{aligned} \phi &= \lambda\phi_1 + \mu\phi_2, & \phi' &= \lambda\phi_1' + \mu\phi_2', \\ \phi'' &= \lambda\phi_1'' + \mu\phi_2'', & \phi''' &= \lambda\phi_1''' + \mu\phi_2'''. \end{aligned} \tag{11}$$

The constants λ and μ can be eliminated from these equations in four different ways and if this is done then we obtain

$$y_1\phi'' - y_2\phi' + y_4\phi = 0, \tag{12}$$

$$y_1\phi''' - y_3\phi' + y_5\phi = 0, \tag{13}$$

$$y_2\phi''' - y_3\phi'' + y_6\phi = 0, \tag{14}$$

and

$$y_4\phi''' - y_5\phi'' + y_6\phi' = 0. \tag{15}$$

Thus we have four possible equations for the determination of the eigenfunction ϕ.

Consider first the behavior of the solutions of Eqs. (12)–(15) near $x = 0$. For this purpose we observe that as $x \to 0$ we have

$$\begin{aligned} y_1 &\sim \tfrac{1}{12}x^4, & y_2 &\sim \tfrac{1}{3}x^3, & y_3 &\sim \tfrac{1}{2}x^2, \\ y_4 &\sim \tfrac{1}{2}x^2, & y_5 &\sim x, & y_6 &\sim 1, \end{aligned} \tag{16}$$

and this limiting behavior is seen to be independent of the coefficients in (1). The point $x = 0$ is therefore a regular singular point of Eqs. (12)–(15) and at that point they have exponents (2, 3), (−2, 2, 3), (−½, 2, 3), and (0, 2, 3), respectively. It is easy

to show, however, as a consequence of (8), that none of the solutions contains logarithmic terms. Accordingly, near $x = 0$ the solution of Eqs. (12)–(15) that satisfies the boundary conditions must be of the form

$$\phi(x) = \sum_{s=0}^{\infty} b_s x^{s+2}, \tag{17}$$

where b_0 and b_1 are arbitrary. When Eq. (1) is even moderately stiff, however, forward integration of Eqs. (12)–(15) from $x = 0$ to 1 leads, as might have been expected, to a serious growth problem.

To avoid this growth problem, consider the possibility of determining ϕ by integrating one of Eqs. (12)–(15) backwards from $x = 1$ to 0. For illustrative purposes we shall suppose that $\phi'(1) = \phi'''(1) = 0$ as these are the relevant boundary conditions for the Orr–Sommerfeld problem which will be discussed in Section 3.2. In all cases it is convenient to fix the normalization of the solution so that $\phi(1) = 1$ and, for the present discussion, we shall assume that $y_1(x)$ does not vanish anywhere in the interval $0 < x \leqslant 1$. For Eq. (12) the initial conditions are $\phi(1) = 1$ and $\phi'(1) = 0$, and (12) then shows that $\phi'''(1)$ vanishes automatically. On integrating Eq. (12) from $x = 1$ to 0, we see that ϕ must necessarily satisfy the boundary conditions at $x = 0$ since the exponents of (12) at $x = 0$ are 2 and 3. Thus we have a simple marching problem for the determination of ϕ. This procedure, however, will fail to yield the final values at $x = 0$ since (12) is singular there, but this is only a very minor limitation.

For Eqs. (13)–(15) the initial conditions are $\phi(1) = 1$, $\phi'(1) = 0$, and $\phi''(1) = -y_4(1)/y_1(1)$. On integrating these equations from $x = 1$ to 0, however, some numerical difficulties would be expected due to the exponents -2, $-\frac{1}{2}$, and 0 of the equations, respectively, at $x = 0$. These difficulties, which are particularly severe in the case of (13), have been confirmed by actual calculations on the Orr–Sommerfeld problem.

It remains to be shown, however, that the solution of Eq. (12) obtained in this way is also a solution of Eq. (1). For this purpose let

$$z_1 = y_1\phi'' - y_2\phi' + y_4\phi, \tag{18}$$

$$z_2 = y_1\phi''' - y_3\phi' + y_5\phi, \tag{19}$$

$$z_3 = y_2\phi''' - y_3\phi'' + y_6\phi, \tag{20}$$

and

$$z_4 = y_4\phi''' - y_5\phi'' + y_6\phi'. \tag{21}$$

A short calculation then shows that

$$z_1' = z_2\,, \tag{22}$$

$$z_2' = a_2 z_1 + a_1 z_2 + z_3 + y_1 L(\phi), \tag{23}$$

$$z_3' = -a_3 z_1 + a_1 z_3 + z_4 + y_2 L(\phi), \tag{24}$$

and

$$z_4' = a_4 z_1 + a_1 z_4 + y_4 L(\phi). \tag{25}$$

Suppose now that (12) holds, i.e., that $z_1 = 0$. Then $z_2 = 0$ from (22) and Eqs. (23)–(25) reduce to

$$0 = z_3 + y_1 L(\phi), \tag{26}$$

$$z_3' = a_1 z_3 + z_4 + y_2 L(\phi), \tag{27}$$

and

$$z_4' = a_1 z_4 + y_4 L(\phi). \tag{28}$$

With ϕ determined from (12) as described above, it can then be shown from (20) and (21) that $z_3(1) = z_4(1) = 0$. Elimination of $L(\phi)$ from Eqs. (26)–(28) then gives a pair of first-order equations for z_3 and z_4, and the only solution of these equations which satisfies the initial conditions is the trivial one. Thus $L(\phi) = 0$ and the solution of (12) is indeed the required eigenfunction.

An alternative to the method just described would be to define the solution matrix (5) with respect to the boundary condition at $x = 1$. In this approach, with $\phi'(1) = \phi'''(1) = 0$, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ would be required to satisfy the initial conditions

$$\boldsymbol{\phi}_1(1) = [1, 0, 0, 0]^T \quad \text{and} \quad \boldsymbol{\phi}_2(1) = [0, 0, 1, 0]^T. \tag{29}$$

The eigenvalue of the problem is then obtained by integrating (8) backwards, subject to the initial condition

$$\mathbf{y}(1) = [0, 1, 0, 0, 0, 0]^T, \tag{30}$$

and by requiring that $y_1(0) = 0$. We also note that as $x \to 1$ we have

$$\begin{aligned} &y_1 \sim x - 1, && y_2 \sim 1, && y_3 \sim a_2(1)(x-1), \\ &y_4 \sim -\tfrac{1}{3}a_4(1)(x-1)^3, && y_5 \sim -a_4(1)(x-1)^2, && y_6 \sim -a_4(1)(x-1), \end{aligned} \tag{31}$$

and hence $x = 1$ is a regular singular point of (12) and (15) but it is an ordinary point of (13) and (14). To obtain the eigenfunction we must now integrate forward from $x = 0$ to 1 to avoid the growth problem and, for this purpose, Eq. (14) would appear to be the best choice since $x = 0$ and 1 are both ordinary points of this equation. The initial conditions are $\phi(0) = \phi'(0) = 0$ and $\phi''(0) = 1$ (say). An argument similar to the one given above then shows that the solution obtained in this way is also a solution of (1) and that ϕ' and ϕ''' both vanish automatically at $x = 1$. The eigenfunction ϕ can then be renormalized, if desired, so that $\phi(1) = 1$.

3. Examples

3.1. *The Transverse Vibrations of a Beam*

To illustrate some features of the method just described, consider first the transverse vibrations of a beam which are governed by the equation

$$\phi^{iv} - k^4\phi = 0 \tag{32}$$

and suppose that one end of the beam, at $x = 0$ (say), is clamped, i.e., $\phi(0) = \phi'(0) = 0$. Then the solutions of (32) which satisfy the initial conditions (4) are

$$\phi_1(x) = \tfrac{1}{2}k^{-2}(\cosh kx - \cos kx)$$

and (33)

$$\phi_2(x) = \tfrac{1}{2}k^{-3}(\sinh kx - \sin kx).$$

There is, of course, no difficulty in solving this problem analytically subject to appropriate boundary conditions at $x = 1$ but there is clearly a growth problem numerically due to the fact that, when kx is large, the solutions (33) no longer form a "numerically satisfactory" pair in the sense of Miller [6].

For this problem $a_1 = a_2 = a_3 = 0$ and $a_4 = k^4$, and the elements of $\mathbf{y}$ then satisfy the equations

$$\begin{aligned} &y_1' = y_2\,, \quad y_2' = y_3 + y_4\,, \quad y_3' = y_5\,, \\ &y_4' = y_5\,, \quad y_5' = -k^4y_1 + y_6\,, \quad y_6' = -k^4y_2\,. \end{aligned} \tag{34}$$

These equations admit the two exact relations

$$k^4y_1 + y_6 = C_1 \quad \text{and} \quad y_3 - y_4 = C_2\,, \tag{35}$$

where C_1 and C_2 are arbitrary constants. The initial conditions (10), however, show that $C_1 = 1$ and $C_2 = 0$ and hence that

$$k^4y_1 + y_6 = 1 \quad \text{and} \quad y_3 - y_4 = 0. \tag{36}$$

The system of equations (34) can therefore be reduced to one of order four. For this purpose let

$$Y_1 = -k^4y_1 + y_6\,, \quad Y_2 = y_2\,, \quad Y_3 = y_3\,, \quad \text{and} \quad Y_4 = y_5\,. \tag{37}$$

Then $\mathbf{Y} = [Y_1\,, Y_2\,, Y_3\,, Y_4]^T$ satisfies the equation

$$\mathbf{Y}^{iv} + 4k^4\mathbf{Y} = 0, \tag{38}$$

and the solution of this equation which satisfies the initial condition $\mathbf{Y}(0) = [1, 0, 0, 0]^T$ is

$$\begin{aligned} Y_1(x) &= \cosh kx \cos kx, \\ Y_2(x) &= k^{-3}(\cosh kx \sin kx - \sinh kx \cos kx), \\ Y_3(x) &= k^{-2} \sinh kx \sin kx, \end{aligned} \tag{39}$$

and

$$Y_4(x) = k^{-1}(\cosh kx \sin kx + \sinh kx \cos kx).$$

When kx is large, these components of $\mathbf{Y}$ have the common growth factor e^{kx}. Nevertheless they remain numerically satisfactory because they differ in phase by the largest possible factors and this is essential if we wish to be able to distinguish different boundary conditions at $x = 1$.

If, for example, we now suppose that the beam is also clamped at $x = 1$, then we must require that $y_1(1) = 0$. This condition is equivalent to requiring that $Y_1(1) = 1$ and hence we obtain the familiar eigenvalue relation $\cosh k \cos k = 1$. For the lowest mode $k = 4.7300407$ and the behavior of the elements of $\mathbf{y}$ for this mode, obtained by numerical integration of (34), is shown in Fig. 1a. To obtain the eigen-

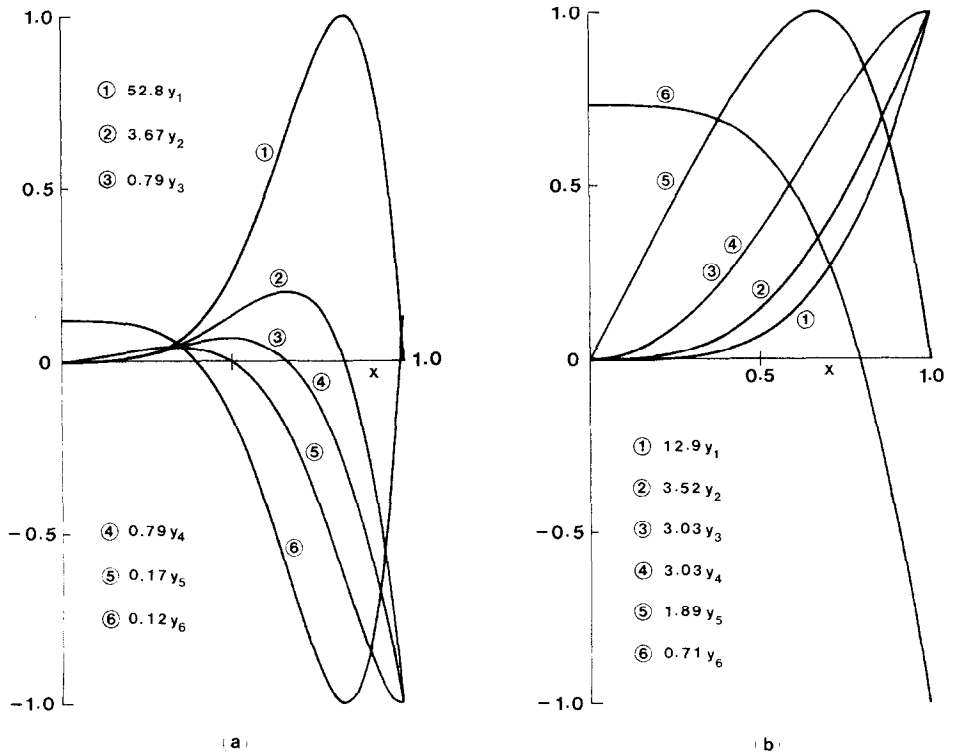

FIG. 1. The behavior of the elements of $\mathbf{y}$ for the vibrating beam problem with $\phi(0) = \phi'(0) = 0$. For (a) $\phi(1) = \phi'(1) = 0$ and for (b) $\phi'(1) = \phi'''(1) = 0$.

function for this problem, however, a slight modification of the procedure described in Section 2 is required since $y_1(1) = 0$. In this case the integration can be initiated at $x = 1$ by using either (14) or (15) and then switching to (12) to continue the integration to $x = 0$.

This problem has been widely used [8, 9, 11] to illustrate the use of the Riccati transformation method and it is of some interest therefore to consider briefly the relationship between that method and the present one. The Riccati method is successful in overcoming the growth problem but other difficulties arise due to the singularities of the Riccati matrix **R** and its inverse **S**. If, however, an attempt is made to eliminate the singularities from the Riccati method then, as Davey [1] has shown, one is led directly to the use of compound matrices as described in this paper. He has also shown how the elements of **R** and **S** can be expressed in terms of the elements of **y** and from those results it follows that

$$\det \mathbf{R} = y_1/y_6 \qquad \text{and} \qquad \det \mathbf{S} = y_6/y_1 \,. \tag{40}$$

Thus the singularities of **R** and **S** occur at the zeros of y_6 and y_1, respectively. As can be seen in Fig. 1a, y_1 vanishes only at the end points $x = 0$ and 1 but y_6 vanishes at $x = 0.3964$ and $x = 0.9924$, and the closeness of the singularities of **R** and **S** at $x = 0.9924$ and 1, respectively, clearly leads to further difficulties in the switching process. To cope with these switching problems, Sloan [8] introduced a free parameter λ and a modified Riccatti matrix **E** (with inverse **G**) such that

$$\det \mathbf{E} = (y_1 - \lambda y_4)/y_6 \qquad \text{and} \qquad \det \mathbf{G} = y_6/(y_1 - \lambda y_4). \tag{41}$$

Although no prescription was given for choosing λ, a trial-and-error approach led Sloan to adopt the value $\lambda = -10$. For this value of λ the singularities of **G** are at $x = 0$ and 0.6649 and they are therefore well separated from the singularities of **E**. Thus, although the Riccati method has some attractive features, it would appear that it also has some inherent disadvantages which are absent from the present method.

For other boundary conditions at $x = 1$ we need only require the vanishing of the appropriate element (or linear combination of elements) of **y**. For comparison with the Orr–Sommerfeld problem, we have also considered the case when $\phi'(1) = \phi'''(1) = 0$ and hence y_5 must vanish at $x = 1$. This corresponds to considering the symmetric vibrations of a beam of length 2. The eigenvalue relation for this problem is $\tanh k + \tan k = 0$ and for the lowest mode $k = 2.3650204$. The behavior of the elements of **y** for this mode is shown in Fig. 1b.

3.2. *The Orr–Sommerfeld Problem for Plane Poiseuille Flow*

For the Orr–Sommerfeld equation, the coefficients in Eq. (1) are

$$\begin{aligned} a_1 &= 0, \qquad a_2 = 2\alpha^2 + i\alpha R(U - c), \\ a_3 &= 0, \qquad a_4 = -\{\alpha^4 + i\alpha R[\alpha^2(U - c) + U'']\}, \end{aligned} \tag{42}$$

where $U(x)$ is the basic velocity distribution, α and R are real parameters, and c is the (possibly complex) eigenvalue parameter. For plane Poiseuille flow on the interval $0 \leqslant x \leqslant 2$ we have $U(x) = x(2 - x)$ and if we consider only the symmetric modes then the problem can be studied on the interval $0 \leqslant x \leqslant 1$ with boundary conditions

$$\phi(0) = \phi'(0) = 0 \qquad \text{and} \qquad \phi'(1) = \phi'''(1) = 0, \tag{43}$$

and the general method discussed in Section 2 is thus directly applicable.

To test the effectiveness of the present method on this problem we have considered the unstable mode for $\alpha = 1$ and $R = 10{,}000$ as this is a case for which a comparison can be made with various existing results. In these calculations our aim was not to achieve great accuracy but rather to show that reasonable accuracy can be obtained without difficulty. The calculations were made therefore by using the Runge–Kutta–Gill procedure with constant step-size and they were performed in single-precision arithmetic on a CDC-6600 computer. Thus in Table I we show the effect of step-size

TABLE I

The Effect of Step-Size on the Eigenvalue for the Orr–Sommerfeld Problem with $\alpha = 1$ and $R = 10{,}000$

Number of steps	c
500	0.2375 221 + 0.0037 409 i
600	243 402 i
700	253 400 i
800	258 398 i
900	260 398 i
1000	262 397 i
1100	263 397 i
1200	0.2375 263 + 0.0037 397 i
Thomas [10]	0.2375 259 + 0.0037 404 i
Orszag [7]	0.2375 265 + 0.0037 397 i

on the eigenvalue c and we have also included a comparison with the values obtained by Thomas [10], who used a five-point Numerov finite-difference method with up to 100 grid points, and Orszag [7], who used an expansion in Chebyshev polynomials with up to 50 terms. The behavior of the elements of $\mathbf{y}$ for this problem is shown in Fig. 2 and, in particular, we see that none of them vanishes in the interval (0, 1). Their rapid variation near $x = 1$ is, of course, a direct consequence of the largeness of R. A calculation of the eigenfunction was also made by integrating Eq. (12) backwards from 1 to 0 as described in Section 2, and the effect of step-size on this calculation is given in Table II. These results, together with a more detailed comparison with Thomas's results, show that we can achieve $5D$ accuracy, which is adequate for most

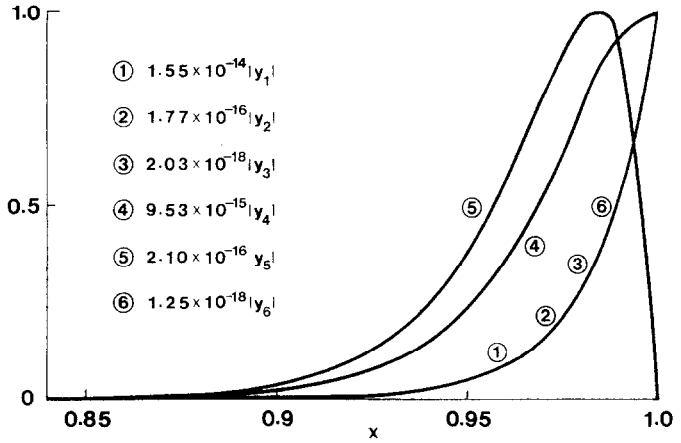

FIG. 2. The behavior of the elements of y for the Orr–Sommerfeld problem. When normalized to unity, the graphs of $|y_1|$, $|y_2|$, $|y_3|$, and $|y_6|$, though not identical, are indistinguishable on this scale.

TABLE II

The Effect of Step-Size on the Eigenfunction
for the Orr–Sommerfeld Problem with $\alpha = 1$ and $R = 10{,}000$

Number of steps	$\phi(0.5)$	$\phi'(0.5)$
500	0.7851 74 − 0.0016 67 *i*	−0.9112 46 − 0.0069 11 *i*
600	81 67 *i*	38 12 *i*
700	84 68 *i*	34 12 *i*
800	85 68 *i*	32 12 *i*
900	86 68 *i*	31 12 *i*
1000	87 68 *i*	31 12 *i*
1100	87 68 *i*	30 12 *i*
1200	0.7851 87 − 0.0016 68 *i*	−0.9112 30 − 0.0069 12 *i*
Thomas [10]	0.7851 90 − 0.0016 62 *i*	
Sloan [8]	0.7851 9 − 0.0016 7 *i*	

purposes, by using about 800 steps. A more systematic comparison of the present method with other existing numerical methods has also recently been made by Gersting [2].

4. DISCUSSION

The relationship between the present method and the Riccati method has been discussed by Davey [1]. It is of some interest, however, to consider briefly its relationship to the method suggested by Golberg [4] which is based on a transformation of the

linear eigenvalue problem into a certain nonlinear initial-value problem. To illustrate the relationship between the two methods, consider the fourth-order system (2) subject to the boundary conditions (43). One of the key steps in relating the two methods is the recognition of the fact that the dependent variables in Golberg's formulation can be identified with $y_1, y_2, y_3, y_4, y_5, y_6$ and this leads to the following system of *five* nonlinear first-order equations:

$$y_5 y_1' = y_1 y_6 + y_3 y_4, \tag{44}$$

$$y_5 y_3' = a_3 y_1 y_5 + a_2(y_1 y_6 + y_3 y_4) + a_1 y_3 y_5 + y_5^2, \tag{45}$$

$$y_5 y_4' = y_5^2, \tag{46}$$

$$y_5 y_5' = (-a_4 y_1 + a_2 y_4 + a_1 y_5 + y_6) y_5, \tag{47}$$

and

$$y_5 y_6' = -a_4(y_1 y_6 + y_3 y_4) - a_3 y_4 y_5 + a_1 y_5 y_6. \tag{48}$$

The other key step then involves the use of the quadratic identity (7) not only to express the terms involving $y_1 y_6 + y_3 y_4$ in Eqs. (44), (45), and (48) in terms of $y_2 y_5$ but also, on differentiation, to obtain the equation $y_2' = y_3 + y_4$. In this way, therefore, it is possible to pass from the fifth-order nonlinear system of equations of Golberg to the sixth-order linear system (8).

In the previous two sections we considered only very simple boundary conditions at $x = 0$ and 1. As Sloan and Wilks [9] have shown, however, the case of general separated boundary conditions can be reduced, without loss of generality, to requiring that any two components of $\boldsymbol{\phi}$ vanish at $x = 0$ (say) and

$$\mathbf{Q}\boldsymbol{\phi}(1) = 0, \qquad \text{where} \qquad \mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \end{bmatrix}. \tag{49}$$

The eigenvalue relation in this more general case then becomes

$$\begin{aligned} &(q_{11}q_{22} - q_{12}q_{21})\, y_1(1) + (q_{11}q_{23} - q_{13}q_{21})\, y_2(1) \\ &\quad + (q_{11}q_{24} - q_{14}q_{21})\, y_3(1) + (q_{12}q_{23} - q_{13}q_{22})\, y_4(1) \\ &\quad + (q_{12}q_{24} - q_{14}q_{22})\, y_5(1) + (q_{13}q_{24} - q_{14}q_{23})\, y_6(1) = 0. \end{aligned} \tag{50}$$

Our interest in the use of compound matrices was stimulated initially by the need to overcome certain difficulties which arose in the asymptotic theory of the eigenvalue relation for the Orr–Sommerfeld problem [5]. It soon became evident, however, that they also provide a simple and effective method for the numerical treatment of eigenvalue problems for stiff differential equations, especially those of hydrodynamic type which are typically of order four or six. There are many aspects of the method which clearly require further study both analytically and numerically. In this paper, for example, we have defined $\mathbf{y}$ as the second compound of the solution matrix $\boldsymbol{\Phi}$ which satisfies certain prescribed initial conditions. More generally, however, if ϕ_1 and ϕ_2

are *any* two solutions of (1) then **y** still satisfies (8). This latter point of view is particularly relevant to the asymptotic theory of the Orr–Sommerfeld problem. While this paper was under review, we also found that compound matrices can be used for linear two-point boundary-value problems. This requires some extension of the ideas presented in this paper and is currently under active study.

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